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Boundary-value problems of the asymmetric theory of elasticity for thin plates☆

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ABSTRACT

Boundary-value problems of the three-dimensional asymmetric micropolar, moment theory of elasticity with free rotation are considered for thin plates. It is assumed that the total stress-strain state is the sum of the internal stress-strain state and the boundary layers, which are determined in an approximation using asymptotic analysis. Three different asymptotic forms are constructed for the three-dimensional boundary-value problem posed, depending on the values of dimensionless physical constants of the plate material. The initial approximation for the first asymptotic form leads to a theory of micropolar plates with free rotation, the initial approximation for the second asymptotic form leads to a theory of micropolar plates with constrained rotation, and the initial approximation for the third asymptotic form leads to a theory of micropolar plates with "small shear stiffness." The corresponding micropolar boundary layers are constructed and studied. The regions of applicability of each of the theories of micropolar plates constructed are indicated.

Interest in the asymmetric micropolar moment theory of elasticity has recently increased owing to new problems in modern micromechanics and nanotechnology.[1–11] An extensive class of two-dimensional problems related to the formulation of a general asymmetric theory of elasticity was previously studied, exact analytic solutions were constructed,[12,13] and problems of wave propagation within the model of a Cosserat solid were considered in Refs.14–16. Another class of problems consists of micropolar beams, plates and shells. Theories of micropolar beams, plates, and shells with constrained rotation have been proposed using the method of hypotheses.[17–19] Theories of thin beams, plates and shells based on the theory of Cosserat surfaces have been developed.[20–22] Alternatives for constructing a theory of plates and shells[23–25] based on the three-dimensional equations of the asymmetric theory of elasticity with independent displacement and rotation fields have been proposed. The fundamental tenets of the general asymmetric theory of elasticity and refined theories of plates and shells[27,28] have been integrated to construct a theory of micropolar shells and plates.[26]

Asymptotic analysis has been used extensively to construct a general theory of thin beams, plates and shells.[29–34] Asymptotic analysis was first used[35] to derive equations that describe the bending of thin plates in displacements and independent rotations from the three-dimensional equations of the asymmetric theory of elasticity. The problem of constructing a general theory of micropolar plates using asymptotic analysis was formulated in Ref. 36.

1. Statement of the problem

Consider an isotropic plate of constant thickness $2h$ as a three-dimensional micropolar elastic solid. We introduce a Cartesian system of coordinates Ox_1, x_2, x_3, so that the Ox_1, x_2 plane coincides with the midplane of the plate. We start out from the fundamental equations of the three-dimensional static problem of the linear asymmetric theory of elasticity with independent displacement and rotation fields:[37–39] the equilibrium equations

$$\sigma_{ji,j} = 0, \quad \mu_{ji,j} + \text{э}_{ijk}\sigma_{jk} = 0 \tag{1.1}$$

the physical relations

$$\sigma_{ji} = (\mu+\alpha)\gamma_{ji} + (\mu-\alpha)\gamma_{ij} + \lambda\gamma_{kk}\delta_{ij},$$
$$\mu_{ji} = (\gamma+\varepsilon)\chi_{ji} + (\gamma-\varepsilon)\chi_{ij} + \beta\chi_{kk}\delta_{ij} \tag{1.2}$$

and the geometric relations

$$\gamma_{ij} = u_{j,i} - \text{э}_{kij}\omega_k, \quad \chi_{ij} = \omega_{j,i} \tag{1.3}$$

Here σ_{ij} and μ_{ij} are the components of the asymmetric force and moment stress tensors, γ_{ij} are the components of the asymmetric strain tensor, χ_{ij} are the component of the asymmetric bend–twist tensor, λ, μ, α, β, γ and ε are the constants of elasticity of the micropolar material, u_i are the components of the displacement

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vector, ω_i are the components of the independent rotation vector at points in the body, $\ni_{kij}$ are the Levi-Civita symbols and the subscripts i, j after a comma denote differentiation with respect to the coordinates x_i, x_j, respectively. Here and henceforth, the subscripts i, j, k take the values 1, 2, 3.

The following force and moment boundary conditions are specified on the faces of the plate $x_3 = \pm h$

$$\sigma_{3i} = p_i^{\pm}, \quad \mu_{3i} = m_i^{\pm} \tag{1.4}$$

where the $p_i^{\pm}, m_i^{\pm}$ are the components of the specified external forces and moments.

The following mixed boundary conditions are specified on the edge of the plate ($\Sigma = \Sigma_1 \cup \Sigma_2$):

$$\sigma_{ji}n_j = p_i^*, \quad \mu_{ji}n_j = m_i^* \text{ on } \Sigma_1; \quad u_i = u_i^*, \quad \omega_i = \omega_i^* \text{ on } \Sigma_2 \tag{1.5}$$

where the p_i^*, m_i^* are the components of the given external forces and moments, n_i are the components of the vector of the normal to the edge surface of the plate, and u_i^*, ω_i^* are the given components of the displacement and independent rotation vectors.

The solution of boundary-value problem (1.1)–(1.5) is the sum of the solutions of the problems that are symmetric and inversely symmetric about x_3. In the symmetric problem $\sigma_{nm}, \sigma_{33}, \mu_{n3}, \mu_{3n}, u_n$ and ω_3 are even functions in x_3, and $\sigma_{n3}, \sigma_{3n}, \mu_{nm}, \mu_{33}, u_3$ and ω_n are odd functions. In the inversely symmetric problem the opposite is true. Here and henceforth, the subscripts n, m take the values 1, 2, and $n \neq m$.

It is assumed that the thickness of the plate $2h$ is small compared with its characteristic dimension a, i.e., $2h \ll a$, $\delta = h/a \ll 1$, and δ is a fundamental small parameter of the problem.

After performing scaling along the coordinate lines in the initial equations and boundary conditions (1.1)–(1.5), as is done in asymptotic analysis, we obtain multipliers with the small parameter δ in front of some derivatives. It is assumed that the full stress-strain state of the thin three-dimensional elastic body forming the plate consists of the internal stress-strain state that extends throughout the entire plate and the boundary layers localized near the edge surface.

2. Introduction of dimensionless coordinates and dimensionless physical parameters

We will change to dimensionless coordinates in Eqs. (1.1)–(1.3) using the formulae

$$\xi = \frac{x_1}{a}, \quad \eta = \frac{x_2}{a}, \quad \zeta = \frac{x_3}{h} \tag{2.1}$$

As a result, we obtain a singularly perturbed system of differential equations with the small parameter δ.

When the asymptotic representations of the boundary-value problem (1.1)–(1.5) are constructed in a thin region of the plate, the values of the physical constants of the micropolar material of the plate play a major role. We introduce the dimensionless parameters

$$\frac{\alpha}{\mu}, \quad \frac{\beta}{a^2\mu}, \quad \frac{\gamma}{a^2\mu}, \quad \frac{\varepsilon}{a^2\mu} \tag{2.2}$$

We will represent the solution of the internal problem in the form of the asymptotic expansion

$$Q = \delta^{-q}\sum_{s=0}^{S}\delta^s Q^{(s)} \tag{2.3}$$

where s is the number of the asymptotic approximation, Q is any of the stresses (force and moment stresses), displacements and independent rotations, q is a natural number, which is different for different quantities and is determined from the condition of self-consistency of the recurrent system of equations after substituting expansion (2.3) into the transformed system (1.1)–(1.3) and equating the coefficients of all the powers of δ to zero in each equation.

Depending on the order of magnitude of the dimensionless physical constants (2.2), we consider three different asymptotic approximations.

3. The theory of micropolar plates with independent displacement and rotation fields

Let us assume that all the dimensionless physical constants (2.2) are of the order of unity. In this case, for the number q in the expansions (2.3), we obtain:

for the symmetric problem with respect to x_3 (a generalized plane stress state of the micropolar plate)

$$q = 2 \ \textbf{\textit{for}} \ \sigma_{nn}, \sigma_{mn}, \mu_{m3}, \mu_{3m}, u_m, \omega_3$$
$$q = 1 \ \textbf{\textit{for}} \ \sigma_{m3}, \sigma_{3m}, \mu_{nn}, \mu_{mn}, \mu_{33}, u_3, \omega_m; \quad q = 0 \ \textbf{\textit{for}} \ \sigma_{33} \tag{3.1}$$

for the inversely symmetric problem with respect to x_3 (bending of the micropolar plate)

$$q = 1 \ \text{for} \ \sigma_{m3}, \sigma_{3m}, \mu_{nn}, \mu_{mn}, \mu_{33}, u_3, \omega_m$$
$$q = 0 \ \text{for} \ \sigma_{nn}, \sigma_{mn}, \sigma_{33}, \mu_{m3}, \mu_{3m}, u_m, \omega_3 \tag{3.2}$$

The specific properties of the internal iterative process are as follows.

1°. The system of equations obtained for the approximation with the superscript s allows of integration with respect to the variable ζ, and the corresponding required quantities will vary across the thickness of the plate according to simple laws. In the initial approximation, for the plane stress-state problem we find that the displacements u_m, the independent rotation ω_3, the force stresses σ_{nn}, σ_{mn} and the moment stresses μ_{m3}, μ_{3m} are constants across the thickness of the plate, the displacement u_3, the independent rotations ω_m, the force stresses σ_{3m}, σ_{m3} and the moment stresses μ_{nn}, μ_{mn}, μ_{33} are linear functions of ζ, and the stress σ_{33} is an even quadratic function of ζ. For the bending problem u_3, ω_m, σ_{3m}, σ_{m3}, μ_{nn}, μ_{mn} and μ_{33} are constant across the thickness of the plate, and $u_m, \omega_3, \sigma_{nn}, \sigma_{mn}, \sigma_{33}$, μ_{m3} and μ_{3m} are linear functions of σ.

2°. Instead of the force and moment stresses, we introduce statically equivalent forces, moments and hypermoments, which are also expanded in asymptotic series when expansion (2.3) is taken into account.

For the generalized plane stress-state problem, we introduce the forces T_{nn}, S_{nm}, the moments $L_{m3}, L_{3m}, M_{m3}, M_{3m}$ and the hypermoments Λ_{nn}, Λ_{mn}, Λ_{33}, which are averaged across the thickness of the plate (henceforth, unless stated otherwise, the integration is carried out from $-h$ to h)

$$T_{nn} = \int\sigma_{nn}dx_3, \quad S_{nm} = \int\sigma_{nm}dx_3, \quad L_{m3} = \int\mu_{m3}dx_3, \quad L_{3m} = \int\mu_{3m}dx_3$$
$$M_{m3} = \int\sigma_{m3}x_3dx_3, \quad M_{3m} = \int\sigma_{3m}x_3dx_3$$
$$\Lambda_{nn} = \int\mu_{nn}x_3dx_3, \quad \Lambda_{mn} = \int\mu_{mn}x_3dx_3, \quad \Lambda_{33} = \int\mu_{33}x_3dx_3 \tag{3.3}$$

For the bending problem, we will introduce the forces N_{m3}, N_{3m}, the moments M_{nn}, M_{mn}, M_{33}, L_{nn}, L_{mn}, L_{33} and the hypermoments Λ_{3n}, Λ_{m3}, which are averaged across the thickness of the plate

$$N_{m3} = \int\sigma_{m3}dx_3, \quad N_{3m} = \int\sigma_{3m}dx_3, \quad M_{nn} = \int\sigma_{nn}x_3dx_3,$$
$$M_{mn} = \int\sigma_{mn}x_3dx_3$$
$$L_{nn} = \int\mu_{nn}dx_3, \quad L_{nm} = \int\mu_{mn}dx_3, \quad L_{33} = \int\mu_{33}dx_3$$
$$M_{33} = \int\sigma_{33}x_3dx_3, \quad \Lambda_{m3} = \int\mu_{m3}x_3dx_3, \quad \Lambda_{3m} = \int\mu_{3m}x_3dx_3 \tag{3.4}$$

Note that a governing system of two-dimensional equations for the internal problem can be obtained for each asymptotic approximation with the superscript s.

For values of the dimensionless physical constants (2.2) of the order of unity and the asymptotic approximations (3.1) and (3.2), we obtain two independent systems of equations.

For the generalized plane stressed state problem we have:

the equilibrium equations

$$\frac{\partial T_{nn}}{\partial x_n} + \frac{\partial S_{mn}}{\partial x_m} = -p_n, \quad \frac{\partial L_{13}}{\partial x_1} + \frac{\partial L_{23}}{\partial x_2} + S_{12} - S_{21} = -m_3 \tag{3.5}$$

the elasticity relations

$$T_{nn} = \frac{2Eh}{1-\upsilon^2}(\Gamma_{nn} + \upsilon\Gamma_{mm}), \quad S_{mn} = 2h[(\mu+\alpha)\Gamma_{mn} + (\mu-\alpha)\Gamma_{nm}]$$
$$L_{n3} = 2h\frac{4\gamma\varepsilon}{\gamma+\varepsilon}k_{n3} + \frac{\gamma-\varepsilon}{\gamma+\varepsilon}L_{3n} \tag{3.6}$$

and the geometric relations

$$\Gamma_{nn} = \frac{\partial\upsilon_n}{\partial x_n}, \quad \Gamma_{nm} = \frac{\partial\upsilon_m}{\partial x_n} - (-1)^m\Omega_3, \quad k_{n3} = \frac{\partial\Omega_3}{\partial x_n} \tag{3.7}$$

Here υ_n are the components of the displacements, Ω_3 is the independent rotation at points in the midplane of the plate, Γ_{nn} and Γ_{nm} are the components of the strains, and k_{n3} are the components of the bend–twist at points in the midplane of the plate.

It should be taken into account that

$$p_n = p_n^+ + p_n^-, \quad m_n = m_n^+ - m_n^-, \quad m_3 = m_3^+ + m_3^-, \quad L_{3n} = hm_n \tag{3.8}$$

Note that the system of equations (3.5)–(3.8) is identical to the corresponding system of equations for the plane strain of a micropolar plate with free rotation previously obtained in Refs. 39–42.

If a solution of the system (3.5)–(3.8) is found, the three-dimensional required quantities (the displacement vector components, the independent rotation vector components, and the force and moment stress tensor components) can be determined using the formulae

$$u_n = \upsilon_n(x_1, x_2), \quad u_3 = x_3\beta_3(x_1, x_2), \quad \omega_n = x_3O_n(x_1, x_2),$$
$$\omega_3 = \Omega_3(x_1, x_2)$$
$$\sigma_{nn} = \frac{T_{nn}(x_1, x_2)}{2h}, \quad \sigma_{mn} = \frac{S_{mn}(x_1, x_2)}{2h}, \quad \sigma_{n3} = \frac{3x_3}{2h^3}M_{n3}(x_1, x_2),$$
$$\sigma_{3n} = \frac{3x_3}{2h^3}M_{3n}(x_1, x_2)$$
$$\mu_{n3} = \frac{L_{n3}(x_1, x_2)}{2h}, \quad \mu_{3n} = \frac{L_{3n}(x_1, x_2)}{2h}, \quad \mu_{nn} = \frac{3x_3}{2h^3}\Lambda_{nn}(x_1, x_2),$$
$$\mu_{mn} = \frac{3x_3}{2h^3}\Lambda_{mn}(x_1, x_2)$$
$$\mu_{33} = \frac{3x_3}{2h^3}\Lambda_{33}$$

For the problem of the bending of micropolar plates with free rotation, we have:

the equilibrium equations

$$\frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} = -p_3, \quad \frac{\partial L_{nn}}{\partial x_n} + \frac{\partial L_{mn}}{\partial x_m} + (-1)^m(N_{m3} - N_{3m}) = -m_n \tag{3.9}$$

the elasticity relations

$$N_{n3} = 2h\frac{4\mu\alpha}{\mu+\alpha}\Gamma_{n3} + \frac{\mu-\alpha}{\mu+\alpha}N_{3n}$$
$$L_{mn} = 2h[(\gamma+\varepsilon)k_{nm} + (\gamma-\varepsilon)k_{mn}]$$
$$L_{nn} = 2h\left[\frac{4\gamma(\gamma+\beta)}{2\gamma+\beta}k_{nn} + \frac{2\gamma\beta}{2\gamma+\beta}k_{mm}\right] + \frac{\beta}{2\gamma+\beta}L_{33} \tag{3.10}$$

the geometric relations

$$\Gamma_{n3} = \frac{\partial w}{\partial x_n} + (-1)^m\Omega_m, \quad k_{nn} = \frac{\partial\Omega_n}{\partial x_n}, \quad k_{nm} = \frac{\partial\Omega_m}{\partial x_n} \tag{3.11}$$

Here w is the deflection of the plate, Ω_n are the independent rotations about the x_1 and x_2 axes at points in the midplane of the plate, Γ_{n3} are the strain components, and the k_{nn}, k_{mn} are the components of the bend–twist at points in the mid-plane of the plate. In addition,

$$N_{3n} = hp_n, \quad L_{33} = hm_3$$
$$p_n = p_n^+ - p_n^-, \quad p_3 = p_3^+ + p_3^-, \quad m_n = m_n^+ + m_n^-, \quad m_3 = m_3^+ - m_3^- \tag{3.12}$$

Note that for values of the dimensionless physical parameters (2.2) of the order of unity in the obtained version of the theory of bending of micropolar plates with free rotation (3.9)–(3.12), free rotations and shear strains play the main role. These equations bear some similarity to the equations of the refined Reissner theory of elastic plates.[43]

Substituting expression (3.10) into equalities (3.9), expressing the strains and bend–twists in the obtained equations in terms of the deflection and angles of rotation using expressions (3.11), we obtain a system of three differential equations in $w(x_1, x_2)$ and $\Omega_n(x_1, x_2)$, which can be reduced to two equations: an equation in the displacement $w(x_1, x_2)$, in which the differential operator will be the operator of the classical theory of plate with bending and the flexural stiffness replaced by the quantity $2h(\gamma+\varepsilon)$, and an equation relative in a function of the angles of rotation of the Helmholtz type.

If a solution of the system of equations (3.9)–(3.12) is found, the required quantities in a three-dimensional region of the plate will be defined as follows:

$$u_3 = w(x_1, x_2), \quad \omega_n = \Omega_n(x_1, x_2),$$
$$u_n = x_3\beta_n(x_1, x_2), \quad \omega_3 = x_3 O_3(x_1, x_2)$$
$$\sigma_{n3} = \frac{1}{2h}N_{n3}(x_1, x_2), \quad \sigma_{3n} = \frac{1}{2h}N_{3n}(x_1, x_2)$$
$$\sigma_{nn} = \frac{3x_3}{2h^3}M_{nn}(x_1, x_2), \quad \sigma_{mn} = \frac{3x_3}{2h^3}M_{mn}(x_1, x_2),$$
$$\sigma_{33} = \frac{3x_3}{2h^3}M_{33}(x_1, x_2)$$
$$\mu_{n3} = \frac{3x_3}{2h^3}\Lambda_{n3}(x_1, x_2), \quad \mu_{3n} = \frac{3x_3}{2h^3}\Lambda_{3n}(x_1, x_2)$$
$$\mu_{nn} = \frac{1}{2h}L_{nn}(x_1, x_2), \quad \mu_{mn} = \frac{1}{2h}L_{mn}(x_1, x_2), \quad \mu_{33} = \frac{1}{2h}L_{33}(x_1, x_2)$$

Let us consider the boundary layer for a micropolar plate with free rotation. We will assume that the plate edge surface Σ on which the boundary layer is located is specified by the equation $x_1 = x_{10}$.

To study the boundary layer in the three-dimensional equations (1.1)–(1.3), we make a replacement of independent variables using the formulae

$$x_1 - x_{10} = h\xi_1, \quad x_2 = a\eta, \quad x_3 = h\zeta \tag{3.13}$$

and we assume that the asymptotic order of all required the quantities remains unchanged when differentiated with respect to ξ_1, η, ζ.

We will represent the solution of Eqs. (1.1)–(1.3) thus transformed in the form

$$Q = \sum_{s=0}^{S} \delta^{\chi_R + s} R^{(s)} \tag{3.14}$$

where R is any of the force and moment stresses, dimensionless displacements ($\upsilon_i = u_i/a$) and independent rotations. Since the force and moment inhomogeneous boundary conditions (1.5), which were specified on the faces of the plate $\zeta = \pm 1$, were satisfied when the internal problem was solved, solution (3.14) should satisfy the homogeneous boundary conditions on the planes $\zeta = \pm 1$:

$$\sigma_{31} = \sigma_{32} = \sigma_{33} = 0, \quad \mu_{31} = \mu_{32} = \mu_{33} = 0 \tag{3.15}$$

After substituting expressions (3.14) into the transformed system (1.1)–(1.3), we obtain a self-consistent system of equations for the approximation with the superscript s if asymptotic expansions of the following form are taken for the required quantities:

$$\sigma_{ij} = \delta^{\chi}\sum_{s=0}^{S}\delta^s\sigma_{ij}^{(s)}, \quad \mu_{ij} = \delta^{\chi}\sum_{s=0}^{S}\delta^s\mu_{ij}^{(s)}$$
$$\upsilon_i = \delta^{\chi+1}\sum_{s=0}^{S}\delta^s\upsilon_i^{(s)}, \quad \omega_i = \delta^{\chi+1}\sum_{s=0}^{S}\delta^s\omega_i^{(s)} \tag{3.16}$$

The integer χ characterizes the intensity of the boundary layer and S indictes the number of asymptotic approximations.

The system obtained for the asymptotic approximation with the superscript s can be written in the form of four systems. Two of them have the form

$$\frac{\partial\sigma_{11}^{(s)}}{\partial\xi_1} + \frac{\partial\sigma_{31}^{(s)}}{\partial\zeta} = R_{1\sigma}^{(s-1)}, \quad \frac{\partial\sigma_{13}^{(s)}}{\partial\xi_1} + \frac{\partial\sigma_{33}^{(s)}}{\partial\zeta} = R_{3\sigma}^{(s-1)}$$
$$\frac{\partial\upsilon_1^{(s)}}{\partial\xi_1} = F_{123\sigma}^{(s)}(\mu, \lambda) + R_{4\sigma}^{(s-1)},$$
$$F_{213\sigma}^{(s)} + R_{5\sigma}^{(s-1)} = 0, \quad \frac{\partial\upsilon_3^{(s)}}{\partial\xi_1} = G_{13}^{(s)}(\mu, \alpha) + R_{6\sigma}^{(s-1)}$$
$$\frac{\partial\upsilon_1}{\partial\zeta} = G_{31}^{(s)}(\mu, \alpha) + R_{7\sigma}^{(s-1)} \tag{3.17}$$

$$\frac{\partial\sigma_{12}^{(s)}}{\partial\xi_1} + \frac{\partial\sigma_{32}^{(s)}}{\partial\zeta} = R_{2\sigma}^{(s-1)}$$
$$\frac{\partial\upsilon_2^{(s)}}{\partial\xi_1} = G_{12\sigma}^{(s)} + R_{8\sigma}^{(s-1)}, \quad G_{21\sigma}^{(s)} + R_{9\sigma}^{(s-1)} = 0,$$
$$G_{23\sigma}^{(s)} + R_{10\sigma}^{(s-1)} = 0, \quad \frac{\partial\upsilon_2^{(s)}}{\partial\zeta} = G_{32\sigma}^{(s)} + R_{11\sigma}^{(s-1)} \tag{3.18}$$

Here

$$F_{i,j,k,\sigma}^{(s)}(\mu, \lambda) = \frac{\mu+\lambda}{\mu(2\mu+3\lambda)}\sigma_{ii}^{(s)} - \frac{\lambda}{2\mu(2\mu+3\lambda)}(\sigma_{jj}^{(s)} + \sigma_{kk}^{(s)})$$
$$G_{ij\sigma}^{(s)}(\mu, \alpha) = \frac{\mu+\alpha}{4\mu\alpha}\sigma_{ij}^{(s)} - \frac{\mu-\alpha}{4\mu\alpha}\sigma_{ji}^{(s)} \tag{3.19}$$

Two more systems are obtained as a result of the following replacement of variables

$$(\sigma_{ij}, u_i, R_{k\sigma}, \mu, \lambda, \alpha) \to (\mu_{ij}, \omega_i, R_{k\mu}, \gamma, \beta, \varepsilon) \tag{3.20}$$

Henceforth we shall refer to the last two systems as (3.17), (3.20) and (3.18), (3.20).

The formulae for $R_{k\sigma}^{(s-1)}$ and $R_{k\mu}^{(s-1)}$ are not given here because of their length. When $s = 0$, these functions are identically equal to zero.

Equation (3.17) describe the force plane strain in the (ξ_1, ζ) plane. When replacement (3.20) is made, equations of antiplane strain of the moment type can be obtained from them. Equation (3.18) describe force antiplane strain (torsion). When replacement (3.20) is made, equations of plane strain of the moment type can be obtained from them.

According to the properties of the boundary layer we should find solutions of the systems of equations (3.17), (3.18) and (3.17), (3.18), (3.20) in the half-strip $0 \le \xi_1 < +\infty$, $-1 \le \zeta \le 1$ that satisfy conditions (3.15) on the faces of the plate and decay rapidly with increasing distance from the edge $x_1 = x_{10}$ ($\xi_1 = 0$) inside the plate.

The requirement for the a decaying stress-strain state to exist imposes additional conditions on the boundary values of the force stresses, moment stresses, displacements and independent rotations in the solutions of the boundary layer. These conditions can be derived directly from Eqs. (3.17), (3.18), (3.20). For this purpose, we apply the operators

$$\int_{-1}^{1} d\zeta \int_{0}^{\infty} (\cdot)\, d\xi_1, \quad \int_{-1}^{1} \zeta\, d\zeta \int_{0}^{\infty} (\cdot)\, d\xi_1, \quad \int_{-1}^{1} d\zeta \int_{0}^{\infty} (\cdot)\, \xi_1\, d\xi_1$$

to the corresponding equations.

After some reduction, we obtain the following conditions on the edge $\xi_1 = 0$

$$\int_{-1}^{1}\sigma_{1i}^{(s)}d\zeta = r_i^{(s-1)}, \quad \int_{-1}^{1}u_2^{(s)}d\zeta = \tilde{r}_2^{(s-1)},$$
$$\int_{-1}^{1}\zeta\sigma_{11}^{(s)}d\zeta + \frac{4\mu\alpha}{\mu+\alpha}\frac{1}{\alpha}\int_{-1}^{1}u_3^{(s)}d\zeta = \tilde{r}_3^{(s-1)}$$
$$\int_{-1}^{1}\zeta\sigma_{13}^{(s)}d\zeta + \frac{4\mu(\mu+\lambda)}{\lambda}\frac{1}{a}\int_{-1}^{1}u_1^{(s)}d\zeta = \tilde{r}_1^{(s-1)} \tag{3.21}$$

Another group of conditions can be obtained after making a replacement of variables using (3.20) in Eqs. (3.21). Equalities (3.21) hold both for quantities with the subscript (p) and for quantities with the subscript (a), that are introduced below.

The quantities $r_i^{(s-1)}, \tilde{r}_i^{(s-1)}$ are expressed in terms of the right-hand sides of $R_{k\sigma}^{(s-1)}$ (k = 1, 2, ..., 11) of the governing equations (3.17), (3.18) for the boundary layer. When s = 0, they are equal to zero.

These conditions will play an important role in combining the asymptotic expansions for the internal problem and for the boundary-layer problems when the three-dimensional boundary conditions (1.5) on the edge surface Σ of the plate are satisfied.

The solutions of the systems of equations (3.17), (3.18), (3.20) that decay as $\xi_1 \to \infty$ and satisfy the homogeneous boundary conditions (3.15) on the faces of the plate ($\zeta = \pm 1$) will be called boundary-layer functions.

When s = 0, systems (3.17), (3.18), (3.20) are homogeneous and independent. We shall call the boundary layer described by system of equation (3.17) when s = 0 a pure force plane boundary layer, we shall call the boundary layer described by system (3.17), (3.20) a pure moment antiplane boundary layer, we shall call the boundary layer described by system (3.18) a pure force antiplane boundary layer, and we shall call the boundary layer described the system (3.18), (3.20) a pure moment plane boundary layer.

We find the decaying solution of each of the four boundary-layer problems by separating the variables:

$$R_{(p)}^{(0)} = \sum_{(\lambda_\gamma)}A_{(p)r}^{(0)}\tilde{R}_{(p)r}(\lambda_r,\zeta)\exp(-\lambda_r\xi_1) \quad (A_{(p)r}^{(0)} \leftrightarrow B_{(p)r}^{(0)})(p \to a),$$
$$\mathrm{Re}\lambda_r > 0 \tag{3.22}$$

Here $R_{(p)}^{(0)}(p \to a)$ represents any of the required quantities for a plane (with the subscript (p)), antiplane (with the subscript (a)), force and moment boundary layer, respectively; $A_{(p)r}^{(0)}$ and $B_{(a)r}^{(0)}$ are unknown constants that apply to the force and moment boundary layers, respectively.

For a pure force plane boundary layer, the functions $\tilde{R}_{(p)r}(\lambda_r,\zeta)$ are expressed in terms of the function $\Phi_r(\lambda_r,\zeta)$, for which the following formulae hold:

for the bending problem

$$\Phi_r(\lambda_r,\zeta) = -\left(\mathrm{tg}\lambda_r + \frac{1}{\lambda_r}\frac{b_2}{b_1}\right)\cos\lambda_r\zeta + \zeta\sin\lambda_r\zeta \tag{3.23}$$

where λ_r is the root of the equation

$$k_1\sin 2\lambda_r + 2\lambda_r = 0 \tag{3.24}$$

for the generalized plane stress state problem

$$\Phi_r(\lambda_r,\zeta) = -\left(\mathrm{ctg}\lambda_r - \frac{1}{\lambda_r}\frac{b_2}{b_1}\right)\sin\lambda_r\zeta + \zeta\cos\lambda_r\zeta \tag{3.25}$$

where λ_r is the root of the equation

$$k_1\sin 2\lambda_r - 2\lambda_r = 0 \tag{3.26}$$

Here

$$k_1 = -1 + \frac{2\alpha}{\mu}\left(2+\frac{\lambda}{\mu}\right)\left(\frac{\alpha}{\mu}-\frac{\lambda}{\mu}-1\right)^{-1},$$
$$b_1 = 2\mu, \quad b_2 = \frac{2(\mu+\alpha)(2\mu+\lambda)}{\alpha-\mu-\lambda} \tag{3.27}$$

It can be shown that the following generalized orthogonality condition holds for boundary-value problem (3.17) when s = 0

$$\int_{-1}^{1}(\sigma_{11r(p)}^{(0)}\upsilon_{1\upsilon(p)}^{(0)} - \sigma_{13\upsilon(p)}^{(0)}\upsilon_{3r(p)}^{(0)})d\zeta = 0, \quad r, \upsilon = 1, 2, \ldots \tag{3.28}$$

The equation for a pure moment antiplane boundary layer can be obtained by replacing the required quantities in formulae (3.22)–(3.28) according to (3.20) and replacing $A_{(p)r}^{(0)}$ by $B_{(a)r}^{(0)}$.

For a pure force antiplane boundary layer the functions $\tilde{R}_{(a)r}(\lambda_r,\zeta)$ are expressed in terms of the function $F_r(\lambda_r,\zeta)$, for which the following formulae: for the generalized plane stress state

$$F_r(\lambda_r,\zeta) = \cos\lambda_r\zeta, \quad \lambda_r = r\pi; \quad r = 1, 2, \ldots \tag{3.29}$$

and for the bending problem

$$F_r(\lambda_r,\zeta) = \sin\lambda_r\zeta, \quad \lambda_r = (2r+1)\frac{\pi}{2}; \quad r = 0, 1, 2, \ldots \tag{3.30}$$

For a pure moment plane boundary layer, formulae (3.22), (3.29), (3.30) are valid when the replacement (3.2) and the replacement of $A_{(a)r}^{(0)}$ with $B_{(p)r}^{(0)}$ are taken into account.

When $s \geq 1$, the systems of equations (3.17), (3.18), (3.20) for all four types of boundary layers become inhomogeneous and have right-hand sides that are known from the preceding approximations. The structure of the general solution of the boundary-value problem for each of the four types of boundary layers is defined by the equality

$$R_{(p)}^{(s)} = \sum_{\lambda_r}A_{(p)r}^{(s)}\tilde{R}_{(p)r}^{(s)}(\lambda_r,\zeta)\exp(-\lambda_r\xi_1) + R_{*(p)}^{(s)} + R_{*(a)}^{(s)} \; (A_{(p)r}^{(s)} \leftrightarrow B_{(p)r}^{(s)},$$
$$p \leftrightarrow a) \tag{3.31}$$

The first term represents the general solution of the corresponding homogeneous boundary-value problem and is constructed precisely like the homogeneous solution in the zeroth approximation (s = 0). The second and third terms are particular solutions of the corresponding equations that satisfy homogeneous boundary conditions (3.15). These solutions do not contain undetermined coefficients.

Thus, the differential equations of the internal problem and the boundary layers have been constructed using the asymmetric theory of elasticity with independent displacement and rotation fields. Therefore, the general solution of three-dimensional problem (1.1)–(1.5) can be represented in the form

$$J = \delta^{-q+s}Q_{(int)}^{(s)} + \delta^{\chi+s}R_{(p)}^{(s)} + \delta^{\mu+s}R_{(a)}^{(s)} \tag{3.32}$$

where the integers χ and μ characterize the intensities of the plane and antiplane boundary layers, respectively. They should be selected so that it would be possible to satisfy the three-dimensional boundary conditions (1.5) on the edge surface Σ of the plate. The subscript in parentheses (*int*) indicates that the quantity belongs to the internal stress-strain state. Summation from $s=0$ to $s=S$ is implicit in the repeated index s.

Let us obtain the boundary conditions for the two-dimensional theory and for a boundary layer.

To fix our ideas, we will consider the case in which the following three-dimensional boundary conditions of the first boundary-value problem are specified on the surface Σ ($\xi_1=\xi_{10}$)

$$\sigma_{1i} = p_i^*, \quad \mu_{1i} = m_i^* \tag{3.33}$$

We substitute the expressions (3.32) into the three-dimensional boundary conditions (3.33). We obtain a self-consistent iterative process when $\chi=\mu=-2$ for the generalized plane stress state problem and when $\chi=\mu=-1$ for the bending problem. Conditions (3.21) were used when deriving the boundary conditions for the internal problem and the boundary-layer problems.

In the zeroth asymptotic approximation ($s=0$), taking conditions (3.21) into account, we obtain the boundary conditions of the theory of micropolar plates with free rotation on the edge $x_1=x_{10}$ for the generalized plane stress state

$$T_{11} = \int_{-h}^{h} p_1^* dx_3, \quad S_{12} = \int_{-h}^{h} p_2^* dx_3, \quad L_{13} = \int_{-h}^{h} m_3^* dx_3 \tag{3.34}$$

$$N_{13} = \int_{-h}^{h} p_3^* dx_3, \quad L_{11} = \int_{-h}^{h} m_1^* dx_3, \quad L_{12} = \int_{-h}^{h} m_2^* dx_3 \tag{3.35}$$

Thus, Eqs. (3.5)–(3.7) and boundary conditions (3.34) comprise a model of the theory of the generalized plane stress state of micropolar plates with free rotation, and Eqs. (3.9)–(3.11) and boundary conditions (3.35) comprise a model of the theory of the bending of micropolar plates with free rotation.

On the edge $\xi_1=0$, for the boundary layer we obtain the following boundary conditions:

in the generalized plane stress-state problem:
$\sigma_{11(p)}=f_1(\zeta)$, $\sigma_{13(p)}=0$ for the pure force plane boundary layer
$\sigma_{12(a)}=f_2(\zeta)$ for the pure force antiplane boundary layer
$\mu_{11(a)}=0$, $\mu_{13(a)}=\Psi_3(\xi)$ for the pure moment antiplane boundary layer
$\mu_{12(p)}=0$ for the pure moment plane boundary layer
in the bending problem:
$\sigma_{11(p)}=0$, $\sigma_{13(p)}=f_3(\zeta)$ for the pure force plane boundary layer
$\sigma_{12(a)}=0$ for the pure force antiplane boundary layer
$\mu_{11(a)}=\Psi_1(\zeta)$, $\mu_{13(a)}=0$ for the pure moment antiplane boundary layer
$\mu_{12(p)}=\Psi_2(\zeta)$ for the pure moment plane boundary layer

Here

$$f_i(\zeta) = p_i^* - \frac{1}{2}\int_{-1}^{1} p_i^* d\zeta, \quad \Psi_i(\zeta) = m_i^* - \frac{1}{2}\int_{-1}^{1} m_i^* d\zeta$$

Note that the differential equations for each of the four boundary layers are homogeneous when $s=0$. Then, if the boundary conditions for the boundary layers are homogeneous, the solution is the zero solution.

Thus, a theory of micropolar plates with free rotation has been formulated using an asymptotic approach. For the generalized plane stress state problem, it consists of the system of equations (3.5)–(3.7) and boundary conditions (3.34), and for the bending problem it consists of the system of equations (3.9)–(3.11) and boundary conditions (3.35). It is noteworthy that each of these systems of equations is of the sixth order and has three boundary conditions on the edge.

For $s=0$ separate boundary-value problems were formulated for each of the four boundary layers. The constants of the boundary layers can be determined by, for example, a variational method, the least squares method, etc. The boundary-value problems for the boundary layers can be solved, for example, by a finite-difference method or finite element analysis.

4. The theory of micropolar plates with constrained rotation

We will assume that the dimensionless physical constants of the plate material can be represented in the form

$$\frac{\alpha}{\mu} \sim 1, \quad \frac{\beta}{a^2\mu} \sim \delta^2\beta_*, \quad \frac{\gamma}{a^2\mu} \sim \delta^2\gamma_*, \quad \frac{\varepsilon}{a^2\mu} \sim \delta^2\varepsilon_* \tag{4.1}$$

where β_*, γ_*, ε_* are quantities of the order of unity.

In this case, for boundary-value problem (1.1)–(1.6) an asymptotic representation form that differs from (3.1), (3.2) exists. We will henceforth confine ourselves to the bending problem. In (2.3) we select the following values of q

$$q = 0 \ \text{for} \ \sigma_{33}, \mu_{3m}, \mu_{m3}; \quad q = 1 \ \text{for} \ \sigma_{3m}, \sigma_{m3}, \mu_{nn}, \mu_{mn}, \mu_{33}$$
$$q = 2 \ \text{for} \ \sigma_{nn}, \sigma_{mn}, u_m, \omega_3; \quad q = 3 \ \text{for} \ u_3, \omega_m \tag{4.2}$$

The asymptotic form (4.2) has the following special features.

1°. Rotations at points in the midplane of the plate are expressed in terms of the deflection of the plate at those points.
2°. The system of equations for one group of the quantities sought represents the theory of micropolar plates with constrained rotation. For μ_{33} and μ_{3m} separate differential equations in the transverse coordinate ζ, and the coordinates ξ, η serve as parameters.

Note that the asymptotic form (4.2) for the quantities of the classical theory of elasticity is identical with the asymptotic form of the corresponding problem in the classical theory of elasticity for plates.[31]

When (4.1) is taken into account, the asymptotic form (4.2) leads to the following system of two-dimensional governing equations of the theory of bending of micropolar plates with constrained rotation: the equilibrium equations

$$\frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} = -p_3$$
$$\frac{\partial(L_{1n}-(-1)^m M_{1m})}{\partial x_1} + \frac{\partial(L_{2n}-(-1)^m M_{2m})}{\partial x_2} + (-1)^m N_{m3} = -m_n + (-1)^m h p_m \tag{4.3}$$

and the physical relations

$$M_{nn} = \frac{2Eh^3}{3(1-\upsilon^2)}\left[\frac{\partial\beta_n}{\partial x_n} + \upsilon\frac{\partial\beta_m}{\partial x_m}\right],$$
$$M_{nm} = \frac{2Eh^3}{3(1+\upsilon)}\left(\frac{\partial\beta_m}{\partial x_n} + \frac{\partial\beta_n}{\partial x_m}\right) + (-1)^n h m_3 + \frac{(-1)^{n-1}}{2}L_{33}$$
$$L_{nn} = 4h\gamma\frac{\partial\Omega_n}{\partial x_n} + \frac{\beta}{2\gamma+\beta}L_{33},$$
$$L_{nm} = 2h(\gamma+\varepsilon)\frac{\partial\Omega_m}{\partial x_n} + 2h(\gamma-\varepsilon)\frac{\partial\Omega_n}{\partial x_m} \tag{4.4}$$

Here

$$\Omega_n = -(-1)^m\beta_m = (-1)^m\frac{\partial w}{\partial x_m}$$

$$p_n = \overset{+}{p}_n - \overset{-}{p}_n, \quad p_3 = \overset{+}{p}_3 + \overset{-}{p}_3, \quad m_n = \overset{+}{m}_n + \overset{-}{m}_n \tag{4.5}$$

$$m_3 = \overset{+}{m}_3 - \overset{-}{m}_3, \quad L_{33} = 2hk_1^{-1}m_3\text{th}k_1, \quad k_1 = \sqrt{\frac{4h^2\alpha}{2\gamma+\beta}} \tag{4.6}$$

Using Eqs. (4.3) and (4.4), we obtain the resolvent for the deflection

$$D^*\nabla^2\nabla^2 w = p_3 - h\left(\frac{\partial p_1}{\partial x_1}+\frac{\partial p_2}{\partial x_2}\right)+\frac{\partial m_2}{\partial x_1}-\frac{\partial m_1}{\partial x_2} \tag{4.7}$$

Here

$$D^* = 2h(\gamma+\varepsilon)+\frac{2Eh^3}{3(1-\upsilon^2)}, \quad D = \frac{2Eh^3}{3(1-\upsilon^2)} \tag{4.8}$$

where D is the classical stiffness of the plate and D^* is the flexural stiffness of micropolar plates with constrained rotation.

The resolvent for the bending of micropolar plates with constrained rotation is similar to the Sophie Germain–Lagrange equation of the classical theory of plates. The difference is in the formulae for the stiffness. In addition, terms related to the external moments m_n appear on the right-hand side of Eq. (4.7).We obtain the following boundary-value problems in μ_{33} and μ_{3m}

$$\frac{\partial^2\mu_{33}}{\partial x_3^2}-\frac{k_1^2}{h^2}\mu_{33} = 0, \quad \mu_{33}(x_3 = h) = m_3 \tag{4.9}$$

$$\frac{\partial^2\mu_{3n}}{\partial x_3^2}-\frac{k_2^2}{h^2}\mu_{3n} = f_n(x_1,x_2,x_3),$$

$$\mu_{3n}(x_3 = h) = m_n; \quad k_2 = \sqrt{\frac{h^2}{\gamma+\varepsilon}\frac{4\mu\alpha}{\mu+\alpha}} \tag{4.10}$$

The coordinates x_1, x_2 serve as parameters.

If a solution of Eqs. (4.7), (4.9) and (4.10) is known, the required quantities in a three-dimensional region of the plate are defined as follows:

$$u_3 = w(x_1,x_2), \quad u_n = x_3\beta_n(x_1,x_2), \quad \omega_n = \Omega_n(x_1,x_2)$$

$$\sigma_{nn} = \frac{3x_3}{2h^3}M_{nn}(x_1,x_2), \quad \mu_{nn} = \frac{1}{2h}L_{nn}(x_1,x_2), \quad \mu_{nm} = \frac{1}{2h}L_{nm}(x_1,x_2)$$

The resolvent for the bending of a micropolar plate (4.7) can be obtained by an asymptotic analysis using formulae (2.3) (4.2) directly from the three-dimensional asymmetric theory of elasticity with constrained rotation.[44] The resolvent for bending of micropolar plates (4.7) was previously obtained in Ref. 18 by the method of hypotheses from the three-dimensional asymmetric theory of elasticity with constrained rotation.[44]

An asymptotic form can be chosen for the problem that is symmetric about x_3, and a theory of the generalized plane stress state of micropolar plates with constrained rotation can be constructed in a similar manner. An identical theory was constructed for the case of plane strain using the method of hypotheses in Refs. 42, 45.

To study the boundary stress-strain state of a plate in Eqs. (1.1)–(1.3), we replace the independent variables using formulae (3.13) and take into account relations (4.1) for the dimensionless physical constants. Then, the solution of Eqs. (1.1)–(1.3), transformed in this way, can be represented in the form (3.14). In the asymptotic approximations we obtain a self-consistent system by selecting the following asymptotic representations for the required quantities

$$\sigma_{ij} = \delta^\chi\sum_{s=0}^{S}\delta^s\sigma_{ij}^{(s)}, \quad \mu_{ij} = \delta^{\chi+1}\sum_{s=0}^{S}\delta^s\mu_{ij}^{(s)},$$

$$\upsilon_i = \delta^{\chi+1}\sum_{s=0}^{S}\delta^s\upsilon_i^{(s)}, \quad \omega_i = \delta^\chi\sum_{s=0}^{S}\delta^s\omega_i^{(s)} \tag{4.11}$$

We represent the equations obtained for the asymptotic approximations in the form of two systems

$$\frac{\partial\sigma_{11}^{(s)}}{\partial\xi_1}+\frac{\partial\sigma_{31}^{(s)}}{\partial\zeta} = R_1^{(s-1)}, \quad \frac{\partial\sigma_{13}^{(s)}}{\partial\xi_1}+\frac{\partial\sigma_{33}^{(s)}}{\partial\zeta} = R_3^{(s-1)},$$

$$\frac{\partial\mu_{12}^{(s)}}{\partial\xi_1}+\frac{\partial\mu_{32}^{(s)}}{\partial\zeta}+a(\sigma_{31}^{(s)}-\sigma_{13}^{(s)}) = \tilde{R}_2^{(s-1)}$$

$$\frac{\partial\upsilon_1^{(s)}}{\partial\xi_1} = F_{123\sigma}^{(s)}(\mu,\lambda)+R_5^{(s-1)}, \quad \frac{\partial\upsilon_3^{(s)}}{\partial\zeta} = F_{312\sigma}^{(s)}(\mu,\lambda),$$

$$F_{213\sigma}^{(s)}(\mu,\lambda)+R_7^{(s-1)} = 0$$

$$\frac{\partial\upsilon_1^{(s)}}{\partial\zeta}-\omega_2^{(s)} = G_{31\sigma}^{(s)}(\mu,\alpha), \quad \frac{\partial\upsilon_3^{(s)}}{\partial\xi_1}+\omega_2^{(s)} = G_{13\sigma}^{(s)}(\mu,\alpha)+R_9^{(s-1)},$$

$$\frac{\partial\omega_2^{(s)}}{\partial\xi_1} = \frac{1}{a\mu}G_{12\mu}^{(s)}(\gamma_*,\varepsilon_*)+\tilde{R}_6^{(s-1)} \tag{4.12}$$

$$\frac{\partial\omega_2^{(s)}}{\partial\zeta} = \frac{1}{a\mu}G_{32\mu}^{(s)}(\gamma_*,\varepsilon_*), \quad \frac{1}{a\mu}G_{21\mu}^{(s)}(\gamma_*,\varepsilon_*)+\tilde{R}_4^{(s-1)} = 0,$$

$$\frac{1}{a\mu}G_{23\mu}^{(s)}(\gamma_*,\varepsilon_*)+\tilde{R}_8^{(s-1)} = 0$$

$$\frac{\partial\sigma_{12}^{(s)}}{\partial\xi_1}+\frac{\partial\sigma_{32}^{(s)}}{\partial\zeta} = R_2^{(s-1)}, \quad \frac{\partial\mu_{11}^{(s)}}{\partial\xi_1}+\frac{\partial\mu_{31}^{(s)}}{\partial\zeta}+a(\sigma_{23}^{(s)}-\sigma_{32}^{(s)}) = \tilde{R}_1^{(s-1)}$$

$$\frac{\partial\mu_{13}^{(s)}}{\partial\xi_1}+\frac{\partial\mu_{33}^{(s)}}{\partial\zeta}+a(\sigma_{12}^{(s)}-\sigma_{21}^{(s)}) = \tilde{R}_3^{(s-1)}$$

$$\frac{\partial\upsilon_2^{(s)}}{\partial\xi_1}-\omega_3^{(s)} = G_{12\sigma}^{(s)}(\mu,\alpha)+R_4^{(s-1)}$$

$$\omega_3^{(s)} = G_{21\sigma}^{(s)}(\mu,\alpha)+R_6^{(s-1)}, \quad \frac{\partial\upsilon_2^{(s)}}{\partial\zeta}+\omega_1^{(s)} = G_{32\sigma}^{(s)}(\mu,\alpha),$$

$$-\omega_1^{(s)} = G_{23\sigma}^{(s)}(\mu,\alpha)+R_8^{(s-1)}$$

$$\frac{\partial\omega_1^{(s)}}{\partial\xi_1} = \frac{1}{a\mu}F_{123\mu}^{(s)}(\gamma_*,\beta_*)+\tilde{R}_5^{(s-1)}, \quad \frac{\partial\omega_3^{(s)}}{\partial\zeta} = \frac{1}{a\mu}F_{312\mu}^{(s)}(\gamma_*,\beta_*),$$

$$\frac{\partial\omega_1^{(s)}}{\partial\zeta} = \frac{1}{a\mu}G_{31\mu}^{(s)}(\gamma_*,\varepsilon_*)$$

$$\frac{1}{a\mu}F_{213\mu}^{(s)}(\gamma_*,\beta_*)+\tilde{R}_7^{(s-1)} = 0, \quad \frac{\partial\omega_3^{(s)}}{\partial\xi_1} = \frac{1}{a\mu}G_{13\mu}^{(s)}(\gamma_*,\varepsilon_*)+\tilde{R}_9^{(s-1)} \tag{4.13}$$

The expressions for $R_k^{(s-1)}$ and $\tilde{R}_k^{(s-1)}$ are not given here because of their length. When $s = 0$, they are equal to zero.

The systems of equations (4.12) and (4.13) have a mixed force–moment form. System (4.12) defines a plane force–moment stress-strain state (plane strain), and system (4.13) defines an antiplane force–moment stress-strain state. The requirement for a decaying stress-strain state to exist imposes the following additional conditions on the boundary values of the variables of the asymmetric theory of elasticity

$$\int_{-1}^{1}\sigma_{1i}^{(s)}d\zeta = E_i^{(s-1)};\quad \int_{-1}^{1}\zeta\sigma_{11}^{(s)}d\zeta+\frac{1}{a}\int_{-1}^{1}\mu_{12}^{(s)}d\zeta = \tilde{E}_1^{(s-1)};$$

$$2a\mu\int_{-1}^{1}\upsilon_2^{(s)}d\zeta-\int_{-1}^{1}\mu_{13}^{(s)}d\zeta = \tilde{E}_2^{(s-1)}$$

$$\int_{-1}^{1}\upsilon_1^{(s)}d\zeta+\frac{\lambda}{4\mu(\mu+\lambda)}\int_{-1}^{1}\zeta\sigma_{13}^{(s)}d\zeta = \tilde{E}_3^{(s-1)}$$

$$\int_{-1}^{1}\omega_2^{(s)}d\zeta+\frac{1}{\gamma_*+\varepsilon_*}\left[\frac{4(\lambda+\mu)}{\lambda+2\mu}\int_{-1}^{1}\zeta\upsilon_1^{(s)}d\zeta+\frac{\lambda}{2\mu(\lambda+2\mu)}\int_{-1}^{1}\zeta^2\sigma_{13}^{(s)}d\zeta\right] = \bar{E}^{(s-1)} \tag{4.14}$$

The quantities $E_k^{(s-1)}$, $\tilde{E}_i^{(s-1)}$, $\bar{E}^{(s-1)}$ are expressed in integral form in terms of $R_i^{(s-1)}$ and $\tilde{R}_k^{(s-1)}$. They are all equal to zero when $s = 0$.

In the initial asymptotic approximation ($s = 0$) systems (4.12) and (4.13) separate completely, and the decaying solutions of both systems can be found by a separation of variables. As a result, we can construct functions of the boundary layer and prove the following generalized orthogonality formula for them

$$\int_{-1}^{1}\left(\sigma_{11r(p)}^{(0)}\upsilon_{1\upsilon(p)}^{(0)}-\sigma_{13\upsilon(p)}^{(0)}\upsilon_{3r(p)}^{(0)}+\frac{1}{a}\mu_{12r(p)}^{(0)}\omega_{2\upsilon(p)}^{(0)}\right)d\zeta = 0$$

Investigating interaction of the inner stress-strain state and the boundary layers, we obtain the boundary conditions of the theory of micropolar plates with constrained rotation for the first boundary-value problem when $s = 0$

$$M_{11}+L_{12} = \int_{-h}^{h}(x_3p_1^*+m_2^*)dx_3,$$

$$N_{13}+\frac{\partial}{\partial x_2}(M_{12}-L_{11}) = \int_{-h}^{h}\left[p_3^*+\frac{\partial}{\partial x_2}(x_3p_2^*-m_1^*)\right]dx_3 \tag{4.15}$$

Boundary conditions (4.15) for micropolar plates with constrained rotation were previously obtained by the method of hypotheses.[46]

Thus, the boundary-value problem of the theory of bending of micropolar plates with constrained rotation is described by system of equations (4.3), (4.4) (or the resolvent (4.7) and boundary conditions (4.15)). The corresponding boundary conditions for the boundary-layer problems for systems (4.12) and (4.13) were also obtained when $s = 0$.

5. The theory of micropolar plates with "small shear stiffness"

It was stated above that different asymptotic forms of boundary-value problem (1.1)–(1.6) of the asymmetric theory of elasticity with independent displacement and rotation fields can be constructed, depending on the values of the dimensionless physical constants of the plate material (2.2). In Sections 3 and 4, we constructed and studied two different asymptotic forms, depending on the values of the dimensionless physical constants of the plate material. These asymptotic forms provided basis for constructing two theories of micropolar plates, namely, a theory with free rotation and a theory with constrained rotation, and corresponding boundary-layer theories.

We will construct a third asymptotic form, which differs from the preceding ones, using the following representations for the dimensionless physical constants

$$\frac{\alpha}{\mu}\sim\delta^2\alpha_*,\quad \frac{\beta}{a^2\mu}\sim 1,\quad \frac{\gamma}{a^2\mu}\sim 1,\quad \frac{\varepsilon}{a^2\mu}\sim 1 \tag{5.1}$$

Consider the bending problem. In expansions (2.3), for q we obtain

$$q = 0 \text{ for } \sigma_{33};\quad q = 1 \text{ for } \sigma_{3m}, \sigma_{m3}, \mu_{33}$$
$$q = 2 \text{ for } \sigma_{nn}, \sigma_{mn}, u_m, \mu_{3m}, \mu_{m3}, \omega_3;\quad q = 3 \text{ for } u_3, \omega_m, \mu_{nn}, \mu_{mn} \tag{5.2}$$

With asymptotic representations (5.2) the quantities of "pure moment" origin in the equations of the theory of micropolar plates can be separated, and they form an autonomous system of equations, for which separate boundary conditions are obtained. For the "force" part of the problem we obtain a unique shear theory of plates, in which the angles of rotation are specified by the "pure moment" part of the problem.

We will formulate these systems of equations:

the equilibrium equations

$$\frac{\partial L_{1n}}{\partial x_1}+\frac{\partial L_{2n}}{\partial x_2} = -m_n \tag{5.3}$$

the elasticity relations

$$L_{nm} = 2h[(\gamma+\varepsilon)k_{nm}+(\gamma-\varepsilon)k_{mn}],\quad L_{nn} = \frac{4h\gamma}{2\gamma+\beta}[2(\gamma+\beta)k_{nn}+\beta k_{mm}] \tag{5.4}$$

and the geometric relations

$$k_{nn} = \frac{\partial\Omega_n}{\partial x_n},\quad k_{nm} = \frac{\partial\Omega_m}{\partial x_n} \tag{5.5}$$

Here k_{nn} and k_{nm} are the components of the bend–twist tensor at points in the midplane of the plate and Ω_n are independent rotations about the axes x_n at points in the midplane.

The equations of the "pure force" part of the plate bending problem are:

the equilibrium equations

$$\frac{\partial N_{13}}{\partial x_1}+\frac{\partial N_{23}}{\partial x_2} = -p_3,\quad N_{3n}-\frac{\partial M_{nn}}{\partial x_n}-\frac{\partial H}{\partial x_m} = hp_n \tag{5.6}$$

the physical relations

$$N_{n3} = 2h\cdot 4\alpha\Gamma_{n3}-N_{3n}$$

$$M_{nn} = -\frac{2Eh^3}{3(1-\upsilon^2)}(K_{nn}+\upsilon K_{mm}),$$

$$M_{nm} = H = \frac{4\mu h^3}{3}K_{nm},\quad K_{nm} = K_{mn} \tag{5.7}$$

and the geometric relations

$$\Gamma_{n3} = \frac{\partial w}{\partial x_n}+(-1)^m\Omega_m,\quad K_{nn} = \frac{\partial^2 w}{\partial x_n^2},\quad K_{nm} = -\frac{\partial^2 w}{\partial x_n\partial x_m} \tag{5.8}$$

Here w is the deflection of the plate, Γ_{n3} are the shear strains, and K_{nm} are the flexural-torsional strains of the midplane of the plate.

For the two-dimensional "force" plate bending problem (5.6)–(5.8) we obtain the following resolvent in $w(x_1, x_2)$

$$D\nabla^2\nabla^2 w + 8h\alpha\left[\frac{\partial}{\partial x_1}\left(\frac{\partial w}{\partial x_1} + \Omega_2\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial w}{\partial x_2} - \Omega_1\right)\right] = -p_3 + h\left(\frac{\partial p_1}{\partial x_1} + \frac{\partial p_2}{\partial x_2}\right) \tag{5.9}$$

The theory of micropolar plates (5.6)–(5.9) has been constructed here for the first time taking relations (5.1), (5.2) into account. We call the coefficient $8h\alpha$ the shear moment stiffness, because the physical constant α is also the shear modulus, as is the classical modulus μ. Then, taking relations (5.1) into account, this theory may be treated as the theory of plates "with small shear stiffness."

Taking relations (5.1) into consideration, the asymptotic expansions for the boundary layers has the form

$$\sigma_{ij} = \delta^{\chi}\sum_{s=0}^{S}\delta^s\sigma_{ij}^{(s)}, \quad \mu_{ij} = \delta^{\chi-3}\sum_{s=0}^{S}\delta^s\mu_{ij}^{(s)},$$
$$\upsilon_i = \delta^{\chi+1}\sum_{s=0}^{S}\delta^s\upsilon_1^{(s)}, \quad \omega_i = \delta^{\chi-2}\sum_{s=0}^{S}\delta^s\omega_i^{(s)} \tag{5.10}$$

We will represent the constitutive equations of the boundary layers in the form of the following four systems of equations

$$\frac{\partial\sigma_{11}^{(s)}}{\partial\xi_1} + \frac{\partial\sigma_{31}^{(s)}}{\partial\zeta} = F_1^{(s-1)}, \quad \frac{\partial\sigma_{13}^{(s)}}{\partial\xi_1} + \frac{\partial\sigma_{33}^{(s)}}{\partial\zeta} = F_3^{(s-1)}$$
$$\frac{\partial\upsilon_1^{(s)}}{\partial\xi_1} = F_{123\sigma}^{(s)}(\mu,\lambda) + F_5^{(s-1)},$$
$$F_{213\sigma}^{(s)}(\mu,\lambda) + F_7^{(s-1)} = 0, \quad \frac{\partial\upsilon_3^{(s)}}{\partial\zeta} = F_{312\sigma}^{(s)}(\mu,\lambda)$$
$$\frac{\partial\upsilon_3^{(s)}}{\partial\xi_1} + \frac{\partial\upsilon_1^{(s)}}{\partial\zeta} + F_9^{(s-1)} = \frac{1}{2\mu}(\sigma_{31}^{(s)} + \sigma_{13}^{(s)}), \quad \omega_2^{(s)} + F_{11}^{(s-1)} = \frac{1}{4\mu\alpha_*}(\sigma_{13}^{(s)} - \sigma_{31}^{(s)}) \tag{5.11}$$

$$\frac{\partial\sigma_{12}^{(s)}}{\partial\xi_1} + \frac{\partial\sigma_{32}^{(s)}}{\partial\zeta} = F_2^{(s-1)}$$
$$\frac{\partial\upsilon_2^{(s)}}{\partial\xi_1} + F_4^{(s-1)} = \frac{1}{2\mu}(\sigma_{12}^{(s)} + \sigma_{21}^{(s)}), \quad \frac{\partial\upsilon_2^{(s)}}{\partial\zeta} + F_8^{(s-1)} = \frac{1}{2\mu}(\sigma_{32}^{(s)} + \sigma_{23}^{(s)})$$
$$\omega_3^{(s)} + F_6^{(s-1)} = \frac{1}{4\mu\alpha_*}(\sigma_{21}^{(s)} - \sigma_{12}^{(s)}), \quad \omega_1^{(s)} + F_{10}^{(s-1)} = \frac{1}{4\mu\alpha_*}(\sigma_{32}^{(s)} - \sigma_{23}^{(s)}) \tag{5.12}$$

$$\frac{\partial\mu_{12}^{(s)}}{\partial\xi_1} + \frac{\partial\mu_{32}^{(s)}}{\partial\zeta} = \tilde{F}_2^{(s-1)}$$
$$\frac{\partial\omega_2^{(s)}}{\partial\xi_1} = aG_{12\mu}^{(s)}(\gamma,\varepsilon) + \tilde{F}_4^{(s-1)}, \quad aG_{21\mu}^{(s)}(\gamma,\varepsilon) + \tilde{F}_6^{(s-1)} = 0$$
$$\frac{\partial\omega_2^{(s)}}{\partial\zeta} = aG_{32\mu}^{(s)}(\gamma,\varepsilon), \quad aG_{23\mu}^{(s)}(\gamma,\varepsilon) + \tilde{F}_8^{(s-1)} = 0 \tag{5.13}$$

$$\frac{\partial\mu_{11}^{(s)}}{\partial\xi_1} + \frac{\partial\mu_{31}^{(s)}}{\partial\zeta} = \tilde{F}_1^{(s-1)}, \quad \frac{\partial\mu_{13}^{(s)}}{\partial\xi_1} + \frac{\partial\mu_{33}^{(s)}}{\partial\zeta} = \tilde{F}_3^{(s-1)}$$
$$\frac{\partial\omega_1^{(s)}}{\partial\xi_1} = aF_{123\mu}^{(s)}(\gamma,\beta) + \tilde{F}_5^{(s-1)}, \quad aF_{213\mu}^{(s)}(\gamma,\beta) + \tilde{F}_7^{(s-1)} = 0$$
$$\frac{\partial\omega_3^{(s)}}{\partial\zeta} = aF_{312\mu}^{(s)}(\gamma,\beta), \quad \frac{\partial\omega_3^{(s)}}{\partial\xi_1} = aG_{13\mu}^{(s)}(\gamma,\varepsilon) + \tilde{F}_9^{(s-1)},$$
$$\frac{\partial\omega_1^{(s)}}{\partial\zeta} = aG_{31\mu}^{(s)}(\gamma,\varepsilon) \tag{5.14}$$

The quantities $F_k^{(s-1)}$ and $\tilde{F}_k^{(s-1)}$ are not given here because of their length. They are equal to zero when $s=0$.

Note that the boundary layers constructed differ considerably from the boundary layers obtained using asymptotic expansions (3.16).

The boundary-layer problems for $\xi_1 = 0$ have the following properties, which were obtained from the systems of equations (5.11)–(5.14)

$$\int_{-1}^{1}\sigma_{1t}^{(s)}d\zeta = e_i^{(s-1)}, \quad \int_{-1}^{1}\mu_{1i}^{(s)}d\zeta = t_i^{(s-1)}, \quad \int_{-1}^{1}\zeta\sigma_{11}^{(s)}d\zeta = \tilde{e}_3^{(s-1)}$$
$$\int_{-1}^{1}\zeta\sigma_{13}^{(s)}d\zeta + \frac{4\mu(\mu+\lambda)}{\lambda}\int_{-1}^{1}\upsilon_1^{(s)}d\zeta = \tilde{e}_1^{(s-1)}$$
$$\int_{-1}^{1}\upsilon_2^{(s)}d\zeta + 2\alpha_*\frac{\gamma-\varepsilon}{4\gamma\varepsilon}\frac{4\gamma(\gamma+\beta)}{2\gamma+\beta}\int_{-1}^{1}\omega_1^{(s)}\zeta d\zeta + 2a\alpha_*\frac{\gamma-\varepsilon}{4\gamma\varepsilon}\frac{\beta}{2\gamma+\beta}$$
$$\int_{-1}^{1}\frac{\zeta^2}{2}\mu_{13}^{(s)}d\zeta = \tilde{e}_2^{(s-1)}$$
$$\int_{-1}^{1}\zeta\mu_{11}^{(s)}d\zeta + \frac{4\gamma\varepsilon}{\gamma-\varepsilon}\frac{1}{a}\int_{-1}^{1}\omega_3^{(s)}d\zeta = \tilde{t}_s^{(s-1)}, \quad \int_{-1}^{1}\omega_2^{(s)}d\zeta = \tilde{t}_2^{(s-1)},$$
$$\int_{-1}^{1}\omega_1^{(s)}d\zeta + a\frac{\beta}{4\gamma(\gamma+\beta)}\int_{-1}^{1}\zeta\mu_{13}^{(s)}d\zeta = \tilde{t}_1^{(s-1)} \tag{5.15}$$

Matching the inner stress-strain state and the boundary layers, we obtain the boundary conditions for systems of equations (5.3)–(5.5) and (5.6)–(5.9).

In the case of the first boundary-value problem, for systems of equations (5.3)–(5.5) we have the following boundary conditions on the contour Γ on the midplane of the plate

$$L_{11} = \int_{-h}^{h}m_1^* dx_3, \quad L_{12} = \int_{-h}^{h}m_2^* dx_3 \tag{5.16}$$

and for systems of equations (5.6)–(5.9) we have

$$N_{13} + \frac{\partial M_{12}}{\partial x_2} = \int_{-h}^{h}(p_3^* + p_2^* x_3)dx_3, \quad M_{11} = \int_{-h}^{h}p_1^* x_3 dx_3 \tag{5.17}$$

In this case, for the boundary-layer problems it is significant that the "pure moment" boundary layers (5.13), (5.14) become the zero boundary layers, i.e., they have a lower degree of intensity when $s=0$ (the final results for non-zero boundary layers are not presented here). The "force" boundary layers (5.11), (5.12) are expressed precisely as they would be according to the classical theory.[31] Thus, the "pure moment" boundary-value problem of micropolar plates is described by system of equations (5.3)–(5.5) with boundary conditions (5.16), and the "force" boundary-value problem is described by system of equations (5.6)–(5.9) with boundary conditions (5.17). These two boundary-value problems define a theory of bending of thin micropolar plates with "small shear stiffness."

For the "force" boundary layers we obtain the following boundary conditions when $\xi_1 = 0$: for a plane "force" boundary layer

$$\sigma_{11(p)} = f_1(\zeta), \quad \sigma_{13(p)} = 0 \tag{5.18}$$

and for an antiplane "force" boundary layer

$$\sigma_{12(a)} = p_2^* - \sigma_{12(int)} \tag{5.19}$$

When $s = 0$, the boundary-value problem for micropolar "force" boundary layers is represented by Eqs. (5.11) and boundary conditions (5.18), as well as by Eqs. (5.12) and boundary conditions (5.19).

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